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A bioheat transfer model: Forced convection in a channel occupied by a porous medium with counterflow

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ABSTRACT

An illustrative model for bioheat transfer is developed. An analytical solution is obtained for forced convection in a parallel plate channel occupied by a layered saturated porous medium with counterflow, the dominant feature that distinguishes bioheat transfer from other forms of heat transfer. The case of asymmetrical constant heat-flux boundary conditions is considered and the Brinkman model is employed for the porous medium. It is found that the Nusselt number Nu is zero when the mean velocity is zero, and negative values can be attained.

1. Introduction

In recent years increasing attention has been given to using a porous medium to model bioheat transfer with counterflow ([1–3]). In particular, Nakayama et al. [4,5] have applied volume averaging to obtain a general set of macroscopic equations for countercurrent bioheat transfer between terminal arteries and veins in the circulatory system. Three energy equations are derived, one for each of the arterial blood, venous blood and solid tissue phases. The equations involve coefficients dependent on convection–perfusion parameters and it is not easy to estimate values for these. The set of general equations has not yet been applied to specific problems.

In this paper a radically new approach is taken. We believe that the main feature that distinguishes bioheat transfer from other forms of heat transfer is the counterflow that is involved in the former (for example, a counterflow in arteries and veins). Accordingly our objective has been to investigate basic forced convection in a channel with counterflow using a model with simplified geometry and without regard to physiological details. As far as we are aware no theoretical work on this problem has been published.

Our analysis is in some aspects an extension of the study by Nield and Kuznetsov [6]. Those authors studied forced convection in a channel with permeability and thermal conductivity varying across the channel. The new feature is that the flow direction now also varies across the channel.

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2. Analysis

The domain of interest is shown in Fig. 1. For the steady-state fully-developed situation we have unidirectional flow in the x^*-direction between impermeable boundaries at $y^* = 0$ and $y^* = H$. We assume that the permeability K and the effective thermal conductivity k are functions of y^* only. Then the Brinkman momentum equation is

$$\mu_{\text{eff}} \frac{d^2 u^*}{dy^{*2}} - \frac{\mu}{K} u^* + G = 0, \tag{1}$$

where μ_{eff} is an effective viscosity, μ is the fluid viscosity, K is the permeability, and G is the applied pressure gradient. We suppose that μ_{eff}, K and G take different values in the two layers. Explicitly, we assume that

$$K = K_1, \quad \mu_{\text{eff}} = \mu_{\text{eff}1}, \quad G = \gamma_1 G_{\text{ref}} \quad \text{for } 0 < y^* < \xi H, \tag{2a}$$
$$K = K_2, \quad \mu_{\text{eff}} = \mu_{\text{eff}2}, \quad G = -\gamma_2 G_{\text{ref}} \quad \text{for } \xi H < y^* < H, \tag{2b}$$

where G_{ref} is a reference value.

We define dimensionless variables

$$x = \frac{x^*}{H}, \quad y = \frac{y^*}{H}, \quad u = \frac{\mu u^*}{G_{\text{ref}} H^2}. \tag{3a, b, c}$$

The dimensionless forms of Eq. (1) are

$$M_1 \frac{d^2 u_1}{dy^2} - N_1 u_1 + 1 = 0, \tag{4a}$$

$$M_2 \frac{d^2 u_2}{dy^2} - N_2 u_2 - 1 = 0, \tag{4b}$$

Nomenclature

c_P	specific heat at constant pressure of the fluid
G	applied pressure gradient
H	channel width
k	effective thermal conductivity
K	permeability
M	pressure-gradient modified viscosity ratio, $\mu_{\mathrm{eff}}/(\gamma\mu)$
N	pressure-gradient modified reciprocal Darcy number, $H^2/(\gamma K)$
Nu	Nusselt number defined in Eq. (21)
P, Q, R	quantities defined in Eq. (30)
q''_{w}	wall heat flux
q''_{μ}	mean wall heat flux, $\frac{1}{2}(q''_{\mathrm{w1}}+q''_{\mathrm{w2}})$
T^*	temperature
T_{m}	bulk mean temperature defined in Eq. (11)
T_{w}	wall temperature
$T_{\mathrm{w}\mu}$	mean of the two wall temperatures, $\frac{1}{2}(T_{\mathrm{w1}}+T_{\mathrm{w2}})$
$\widehat{T}$	dimensionless temperature, $\frac{T^*-T_{\mathrm{w}\mu}}{T_{\mathrm{m}}-T_{\mathrm{w}\mu}}$
u	dimensionless filtration velocity, $\mu u^*/G_{\mathrm{ref}}H^2$
u^*	filtration velocity
$\hat{u}$	rescaled dimensionless velocity, u^*/U
$\bar{u}$	dimensionless mean velocity defined in Eq. (15)
U	mean velocity defined in Eq. (11)
x	dimensionless longitudinal coordinate, x^*/H
x^*	longitudinal coordinate
y	dimensionless transverse coordinate, y^*/H
y^*	transverse coordinate

Greek symbols

β	parameter defined in Eq. (28)
γ	parameter controlling applied pressure gradient, $G = \pm\gamma G_{\mathrm{ref}}$
λ	parameter defined in Eq. (8a,b)
μ	fluid viscosity
μ_{eff}	effective viscosity
ξ	position of the interface between the two layers
ρ	fluid density

Subscripts

1	parameters of the first layer, $0 < y^* < \xi H$
2	parameters of the second layer, $\xi H < y^* < H$

where the pressure-gradient modified viscosity ratios M_1, M_2 and the pressure-gradient modified reciprocal Darcy numbers N_1, N_2 are defined by

$$M_1 = \frac{\mu_{\mathrm{eff1}}}{\gamma_1\mu}, \quad M_2 = \frac{\mu_{\mathrm{eff2}}}{\gamma_2\mu}, \quad N_1 = \frac{H^2}{\gamma_1 K_1}, \quad N_2 = \frac{H^2}{\gamma_2 K_2}. \tag{5a, b, c, d}$$

Eqs. (4a,b) must be solved subject to the boundary conditions

$$u_1 = 0 \quad \text{at } y = 0, \tag{6a}$$
$$u_1 = 0 \quad \text{at } y = \xi, \tag{6b}$$
$$u_2 = 0 \quad \text{at } y = \xi, \tag{6c}$$
$$u_2 = 0 \quad \text{at } y = 1. \tag{6d}$$

The solution of Eqs. (4a,b) subject to Eqs. (6a,b,c,d) is

$$u_1 = \frac{1 + e^{\lambda_1\xi} - e^{\lambda_1 y} - e^{\lambda_1(\xi - y)}}{N_1[1 + e^{\lambda_1\xi}]}, \tag{7a}$$
$$u_2 = -\frac{[1 + e^{-\lambda_2(1-\xi)} - e^{-\lambda_2(1-y)} - e^{-\lambda_2(y-\xi)}]}{N_2[1 + e^{-\lambda_2(1-\xi)}]}, \tag{7b}$$

where

$$\lambda_1 = \sqrt{\frac{N_1}{M_1}}, \quad \lambda_2 = \sqrt{\frac{N_2}{M_2}}. \tag{8a, b}$$

In the Darcy limit ($\lambda_1 \to \infty$, $\lambda_2 \to \infty$) one has slug flow,

$$u_1 = 1/N_1, \quad u_2 = -1/N_2. \tag{9a, b}$$

In the clear (of solid material) fluid limit ($\lambda_1 \to 0$, $\lambda_2 \to 0$) one has plane Poiseuille flow,

$$u_1 = \frac{1}{2M_1}[\xi^2 - y^2 - (\xi - y)^2], \tag{10a}$$
$$u_2 = -\frac{1}{2M_2}[(1-\xi)^2 - (1-y)^2 - (y-\xi)^2]. \tag{10b}$$

The mean velocity U and the bulk mean temperature T_{m} are defined by

$$U = \frac{1}{H}\int_0^H u^*\,dy^*, \quad T_{\mathrm{m}} = \frac{1}{HU}\int_0^H u^*T^*\,dy^*. \tag{11}$$

Further dimensionless variables are defined by

$$\hat{u} = \frac{u^*}{U}, \quad \widehat{T} = \frac{T^* - T_{\mathrm{w}\mu}}{T_{\mathrm{m}} - T_{\mathrm{w}\mu}}, \tag{12a, b}$$

where the mean wall temperature $T_{\mathrm{w}\mu}$ is defined by

$$T_{\mathrm{w}\mu} = \frac{1}{2}(T_{\mathrm{w1}} + T_{\mathrm{w2}}). \tag{13}$$

This implies that

$$\hat{u}_1 = \frac{u_1}{\bar{u}}, \quad \hat{u}_2 = \frac{u_2}{\bar{u}}, \tag{14}$$

where

$$\bar{u} = \int_0^\xi u_1\,dy + \int_\xi^1 u_2\,dy = \frac{1}{N_1}\left\{\xi - \frac{2}{\lambda_1}\tanh\left(\frac{\lambda_1\xi}{2}\right)\right\} - \frac{1}{N_2}\left\{1 - \xi - \frac{2}{\lambda_2}\tanh\left(\frac{\lambda_2(1-\xi)}{2}\right)\right\}. \tag{15}$$

The value for the Darcy limit is

$$\bar{u} = \frac{\xi}{N_1} - \frac{1-\xi}{N_2}. \tag{16}$$

The value for the clear fluid limit is

$$\bar{u} = \frac{1}{6}\left(\frac{\xi^3}{M_1} - \frac{(1-\xi)^3}{M_2}\right). \tag{17}$$

Now suppose that the thermal conductivity is given by

$$k = k_1 \quad \text{for } 0 < |y^*| < \xi H, \tag{18a}$$
$$k = k_2 \quad \text{for } \xi H < |y^*| < H, \tag{18b}$$

so that the mean value is given by

$$\bar{k} = \xi k_1 + (1-\xi)k_2. \tag{19}$$

We write

$$\tilde{k}_i = \frac{k_i}{\bar{k}} \quad \text{for } i = 1, 2. \tag{20}$$

Further we define the Nusselt number Nu based on the channel width as

$$Nu = \frac{Hq''_\mu}{\bar{k}(T_{w\mu} - T_m)}, \tag{21}$$

where q''_μ is the mean wall heat flux, defined in terms of Fig. 1 and

$$q''_\mu = \frac{1}{2}(q''_{w1} + q''_{w2}). \tag{22}$$

The reader should note that we have defined Nu in terms of the channel width rather than the hydraulic diameter (twice the channel width).

The steady-state thermal energy equation (for the case of negligible axial conduction and local thermal equilibrium) is

$$u^* \frac{\partial T^*}{\partial x^*} = \frac{k}{\rho c_p} \frac{\partial^2 T^*}{\partial y^{*2}}. \tag{23}$$

The first law of thermodynamics leads to

$$\frac{\partial T^*}{\partial x^*} = \frac{dT_m}{dx^*} = \frac{2q''_\mu}{\rho c_p HU} = \text{constant}. \tag{24}$$

In this case the dimensionless form of the thermal energy equation may be written as

$$\frac{d^2\widehat{T}_1}{dy^2} = -\frac{2Nu\hat{u}_1}{\tilde{k}_1} \quad \text{for } 0 < y < \xi,$$
$$\frac{d^2\widehat{T}_2}{dy^2} = -\frac{2Nu\hat{u}_2}{\tilde{k}_2} \quad \text{for } \xi < y < 1. \tag{25a, b}$$

These equations must now be solved subject to the boundary conditions

$$\widehat{T}_1(0) = \beta, \quad \widehat{T}_2(1) = -\beta, \tag{26a, b}$$

and the matching conditions (for temperature and heat flux)

$$\widehat{T}_1(\xi) = \widehat{T}_2(\xi), \quad \tilde{k}_1 \frac{d\widehat{T}_1}{dy}(\xi) = \tilde{k}_2 \frac{d\widehat{T}_2}{dy}(\xi), \tag{27a, b}$$

where

$$\beta = \frac{T_{w1} - T_{w2}}{2(T_m - T_{w\mu})}. \tag{28}$$

The solution is of the form

$$\widehat{T}_1 = \frac{\beta P_1(y) + NuQ_1(y)}{R_1}, \tag{29a}$$

$$\widehat{T}_2 = \frac{\beta P_2(y) + NuQ_2(y)}{R_2}. \tag{29b}$$

In the Darcy limit the expressions in Eqs. (29a,b) become

$$P_1(y) = N_1N_2\bar{u}[\tilde{k}_1^2(1-\xi) + \tilde{k}_1\tilde{k}_2(\xi - 2y)], \tag{30a}$$

$$P_2(y) = N_1N_2\bar{u}[\tilde{k}_1\tilde{k}_2(1 - 2y + \xi) - \tilde{k}_2^2\xi], \tag{30b}$$

$$Q_1(y) = -\tilde{k}_1N_1(1-\xi)^2y + \tilde{k}_1N_2(1-\xi)y(2\xi - y) + \tilde{k}_2N_2\xi y(\xi - y), \tag{30c}$$

$$Q_2(y) = \tilde{k}_1N_1(1-\xi)(1-y)(\xi - y) + \tilde{k}_2N_1(2\xi^2 - \xi - 2\xi^2y + \xi y^2) + \tilde{k}_2N_2\xi^2(1-y), \tag{30d}$$

$$R_1 = \tilde{k}_1N_1N_2\bar{u}[\tilde{k}_1(1-\xi) + \tilde{k}_2\xi], \tag{30e}$$

$$R_2 = \tilde{k}_2N_1N_2\bar{u}[\tilde{k}_1(1-\xi) + \tilde{k}_2\xi]. \tag{30f}$$

In the clear fluid limit the corresponding expressions are

$$P_1(y) = 6M_1M_2\bar{u}[\tilde{k}_1^2(1-\xi) + \tilde{k}_1\tilde{k}_2(\xi - 2y)], \tag{31a}$$

$$P_2(y) = 6M_1M_2\bar{u}[\tilde{k}_1\tilde{k}_2(1 - 2y + \xi) - \tilde{k}_2^2\xi], \tag{31b}$$

$$Q_1(y) = -\tilde{k}_1M_1(1-\xi)^4y + \tilde{k}_1M_2(1-\xi)(2\xi^3y - 2\xi y^3 + y^4) + \tilde{k}_2M_2(\xi^4y - 2\xi^2y^3 + \xi y^4), \tag{31c}$$

$$Q_2(y) = -\tilde{k}_1M_1(1-\xi)(1-y)(y-\xi)[1 - 3\xi + \xi^2 + (1+\xi)y - y^2] - \tilde{k}_2M_1\xi(1-y)[1 - 4\xi + 6\xi^2 - 2\xi^3 + (1-4\xi)y + (1+2\xi)y^2 - y^3] + \tilde{k}_2M_2\xi^4(1-y), \tag{31d}$$

$$R_1 = 6\tilde{k}_1M_1M_2\bar{u}[\tilde{k}_1(1-\xi) + \tilde{k}_2\xi], \tag{31e}$$

$$R_2 = 6\tilde{k}_2M_1M_2\bar{u}[\tilde{k}_1(1-\xi) + \tilde{k}_2\xi]. \tag{31f}$$

Finally, substitution into the determining compatibility condition

$$\int_0^1 \hat{u}\widehat{T}\,dy = \int_0^\xi \hat{u}_1\widehat{T}_1\,dy + \int_\xi^1 \hat{u}_2\widehat{T}_2\,dy = 1, \tag{32}$$

then yields an expression for the Nusselt number.

We used the Mathematica software package to obtain this expression (which is too complicated to present here for the general case) and to obtain values of Nu for various values of the input parameters.

In the Darcy limit one has

$$Nu = \frac{6\{-\beta\bar{u}(\tilde{k}_1N_2 + \tilde{k}_2N_1)\xi(1-\xi) + N_1N_2\bar{u}^2[\tilde{k}_1(1-\xi) + \tilde{k}_2\xi]\}}{\frac{\tilde{k}_1N_1}{\tilde{k}_2N_2}(1-\xi)^4 + \frac{4N_1}{N_2}\xi(1-\xi)^3 - 6\xi^2(1-\xi)^2 + \frac{4N_2}{N_1}\xi^3(1-\xi) + \frac{\tilde{k}_2N_2}{\tilde{k}_1N_1}\xi^4}, \tag{33}$$

where $\bar{u}$ is given by Eq. (16).

In the clear fluid limit one obtains

$$Nu = \frac{420\{-\beta\bar{u}[\tilde{k}_1M_2\xi^3(1-\xi) + \tilde{k}_2M_1\xi(1-\xi)^3] + 6M_1M_2\bar{u}^2[\tilde{k}_1(1-\xi) + \tilde{k}_2\xi]\}}{17\frac{\tilde{k}_1M_1}{\tilde{k}_2M_2}(1-\xi)^8 + \frac{52M_1}{M_2}\xi(1-\xi)^7 - 70\xi^4(1-\xi)^4 + \frac{52M_2}{M_1}\xi^7(1-\xi) + 17\frac{\tilde{k}_2M_2}{\tilde{k}_1M_1}\xi^8}, \tag{34}$$

where $\bar{u}$ is given by Eq. (17).

It is interesting to compare the above results with those for the situation where u^*_2 is reversed in sign, so that in effect there is no counterflow; that is the flow in the two layers is in parallel instead of being in anti-parallel.

Then one has, in place of Eqs. (7b), (9b), (10b), (15), (16) and (17),

$$u_2 = \frac{[1 + e^{-\lambda_2(1-\xi)} - e^{-\lambda_2(1-y)} - e^{-\lambda_2(y-\xi)}]}{N_2[1 + e^{-\lambda_2(1-\xi)}]}, \tag{7b*}$$

$$u_1 = 1/N_1, \quad u_2 = 1/N_2. \tag{9a, b*}$$

$$u_2 = \frac{1}{2M_2}[(1-\xi)^2 - (1-y)^2 - (y-\xi)^2]. \tag{10, b*}$$

$$\bar{u} = \int_0^\xi u_1\,dy + \int_\xi^1 u_2\,dy = \frac{1}{N_1}\left\{\xi - \frac{2}{\lambda_1}\tanh\left(\frac{\lambda_1\xi}{2}\right)\right\} + \frac{1}{N_2}\left\{1 - \xi - \frac{2}{\lambda_2}\tanh\left(\frac{\lambda_2(1-\xi)}{2}\right)\right\}. \tag{15*}$$

$$\bar{u} = \frac{\xi}{N_1} + \frac{1-\xi}{N_2}. \tag{16*}$$

$$\bar{u} = \frac{1}{6}\left(\frac{\xi^3}{M_1} + \frac{(1-\xi)^3}{M_2}\right) \tag{17*}$$

Eq. (25b) is unchanged in form.

In place of Eq. (33), for the Darcy limit, one has now

$$Nu = \frac{6\{-\beta\bar{u}(\tilde{k}_1 N_2 - \tilde{k}_2 N_1)\xi(1-\xi) + N_1 N_2 \bar{u}^2[\tilde{k}_1(1-\xi)+\tilde{k}_2\xi]\}}{\frac{\tilde{k}_1 N_1}{\tilde{k}_2 N_2}(1-\xi)^4 + \frac{4N_1}{N_2}\xi(1-\xi)^3 + 6\xi^2(1-\xi)^2 + \frac{4N_2}{N_1}\xi^3(1-\xi) + \frac{\tilde{k}_2 N_2}{\tilde{k}_1 N_1}\xi^4}, \quad (33^*)$$

where $\bar{u}$ is given by Eq. (16*)

$$\bar{u} = \frac{\xi}{N_1} + \frac{1-\xi}{N_2}. \quad (16^*)$$

In place of Eq. (34), in the clear fluid limit one now obtains

$$Nu = \frac{420\{-\beta\bar{u}[\tilde{k}_1 M_2 \xi^3(1-\xi) - \tilde{k}_2 M_1 \xi(1-\xi)^3] + 6M_1 M_2 \bar{u}^2[\tilde{k}_1(1-\xi)+\tilde{k}_2\xi]\}}{17\frac{\tilde{k}_1 M_1}{\tilde{k}_2 M_2}(1-\xi)^8 + \frac{52M_1}{M_2}\xi(1-\xi)^7 + 70\xi^4(1-\xi)^4 + \frac{52M_2}{M_1}\xi^7(1-\xi) + 17\frac{\tilde{k}_2 M_2}{\tilde{k}_1 M_1}\xi^8}, \quad (34^*)$$

where $\bar{u}$ is given by Eq. (17*).

Eqs. (33) and (33*) can be combined to give

$$Nu = \frac{6\{-\beta\bar{u}(\tilde{k}_1 N_2 \pm \tilde{k}_2 N_1)\xi(1-\xi) + N_1 N_2 \bar{u}^2[\tilde{k}_1(1-\xi)+\tilde{k}_2\xi]\}}{\frac{\tilde{k}_1 N_1}{\tilde{k}_2 N_2}(1-\xi)^4 + \frac{4N_1}{N_2}\xi(1-\xi)^3 \mp 6\xi^2(1-\xi)^2 + \frac{4N_2}{N_1}\xi^3(1-\xi) + \frac{\tilde{k}_2 N_2}{\tilde{k}_1 N_1}\xi^4}, \quad (33^{**})$$

where $\bar{u}$ is given by

$$\bar{u} = \frac{\xi}{N_1} \mp \frac{1-\xi}{N_2}. \quad (16^{**})$$

Here the upper alternative sign refers to the counterflow situation.

Similarly, Eqs. (34) and (34*) can be combined to give

$$Nu = \frac{420\{-\beta\bar{u}[\tilde{k}_1 M_2 \xi^3(1-\xi) \pm \tilde{k}_2 M_1 \xi(1-\xi)^3] + 6M_1 M_2 \bar{u}^2[\tilde{k}_1(1-\xi)+\tilde{k}_2\xi]\}}{17\frac{\tilde{k}_1 M_1}{\tilde{k}_2 M_2}(1-\xi)^8 + \frac{52M_1}{M_2}\xi(1-\xi)^7 \mp 70\xi^4(1-\xi)^4 + \frac{52M_2}{M_1}\xi^7(1-\xi) + 17\frac{\tilde{k}_2 M_2}{\tilde{k}_1 M_1}\xi^8}, \quad (34^{**})$$

where $\bar{u}$ is given by

$$\bar{u} = \frac{1}{6}\left(\frac{\xi^3}{M_1} \mp \frac{(1-\xi)^3}{M_2}\right). \quad (17^{**})$$

3. Results and discussion

3.1. General considerations

Our main interest is in the effect of replacing parallel flow by counter flow. The prime effect is evident from observation of the various terms in Eqs. (34**) and (17**). The change in sign in the middle term of the denominator of Eq. (34**) from + to − has a minor effect. Other things being equal, the change would increase the value of *Nu* by a small amount. The sign of the value of the denominator stays positive for all values of the parameters. The important effect is a result of changes in sign in Eq. (17**) and the numerator of Eq. (34**). First, consider the case of symmetric heating ($\beta = 0$). Because of the term in $\bar{u}^2$ the expression for *Nu* cannot become negative, but it can become arbitrarily small, and in fact takes the value zero for the special case when $\bar{u}$ becomes zero, that is when the two terms in Eq. (17**) cancel, e.g. when $M_1 = M_2 = 1$ and $\xi = 1/2$, i.e. in the case of an antisymmetric velocity profile. Second, in the case of asymmetric heating ($\beta \neq 0$) the expression for *Nu* can become negative. A negative value of Nu means that the value of $(T_{w\mu} - T_m)$, the difference between the mean wall temperature and the bulk temperature, has a sign opposite to that of q''_μ, the mean wall heat flux into the fluid domain. The negative values arise in the case of strong thermal asymmetry and when the product of β and $\bar{u}$ is positive, so that the more strongly heated boundary (and thus the hotter one) is adjacent to the layer in which the weaker flow occurs, other things being equal. (Note that *N* is a reciprocal Darcy number, so that large *N* corresponds to small velocity.)

Our general solution contains a large number of parameters, so for numerical computations we have to be selective. First, we report results for only the Darcy and clear fluid limits. These results serve as upper and lower bounds for the Nusselt number for the general case and, as the plots presented below show, there are no major qualitative differences between the results for the two cases.

Second, in this paper we present results just for the case of homogeneity of the thermal conductivity, so we take $\tilde{k}_1 = \tilde{k}_2 = 1$. The general effect of thermal heterogeneity can in fact be deduced from an inspection of the expressions in Eqs. (33) and (34). In each expression the denominator is a relatively weak function of $\tilde{k}_2/\tilde{k}_1$; the value is increased by a relatively small percentage as $\tilde{k}_2/\tilde{k}_1$ moves away from unity in either direction. In the numerator of each expression, $\bar{u}$ is obviously independent of $\tilde{k}_2/\tilde{k}_1$, so for symmetric heating (the case where $\beta = 0$) Nu is approximately proportional to $\tilde{k}_1(1-\xi) + \tilde{k}_2\xi$ (which for the case of $\xi = 1/2$ takes the value 2, independent of $\tilde{k}_2/\tilde{k}_1$). For the case of asymmetric heating

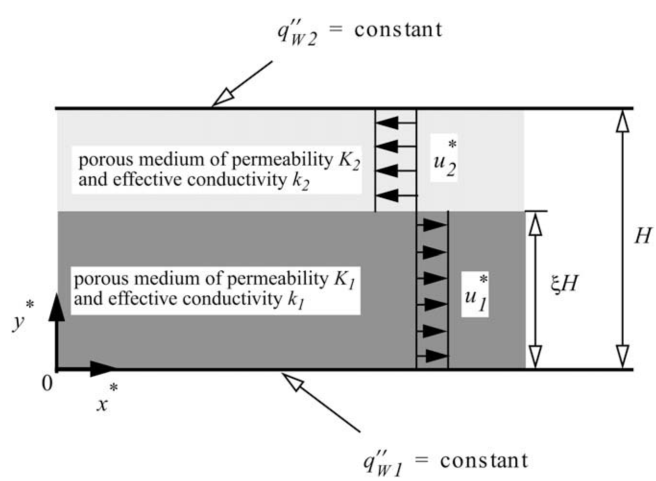

Fig. 1. Definition sketch.

Table 1a
Values of the Nusselt number *Nu* for counterflow in the Darcy limit (Eq. (33))

	$N_2/N_1 = 0.1$	1.0	10.0
$\beta = 0$	4.37	0.00	4.37
1	7.04	0.00	1.70
10	31.07	0.00	−22.33

there is an additional term proportional to $\tilde{k}_1 N_2 + \tilde{k}_2 N_1$ that is involved.

Third, we treat only the case where the layers have equal thickness or the thicknesses are in the ratio 1:9 or 9:1, so we take $\xi = 0.5$, 0.1 and 0.9 in turn, and we concentrate on the first of these cases. That still leaves β and the ratios M_2/M_1 and N_2/N_1 to be varied. It is obvious that Nu varies with β in a linear manner and varies with the other parameters in a more complicated way.

Table 1b
Values of the Nusselt number Nu for parallel flow in the Darcy limit (Eq. (33*))

	$N_2/N_1 = 0.1$	1.0	10.0
$\beta = 0$	5.14	6.00	5.14
1	7.24	6.00	3.04
10	26.17	6.00	−15.89

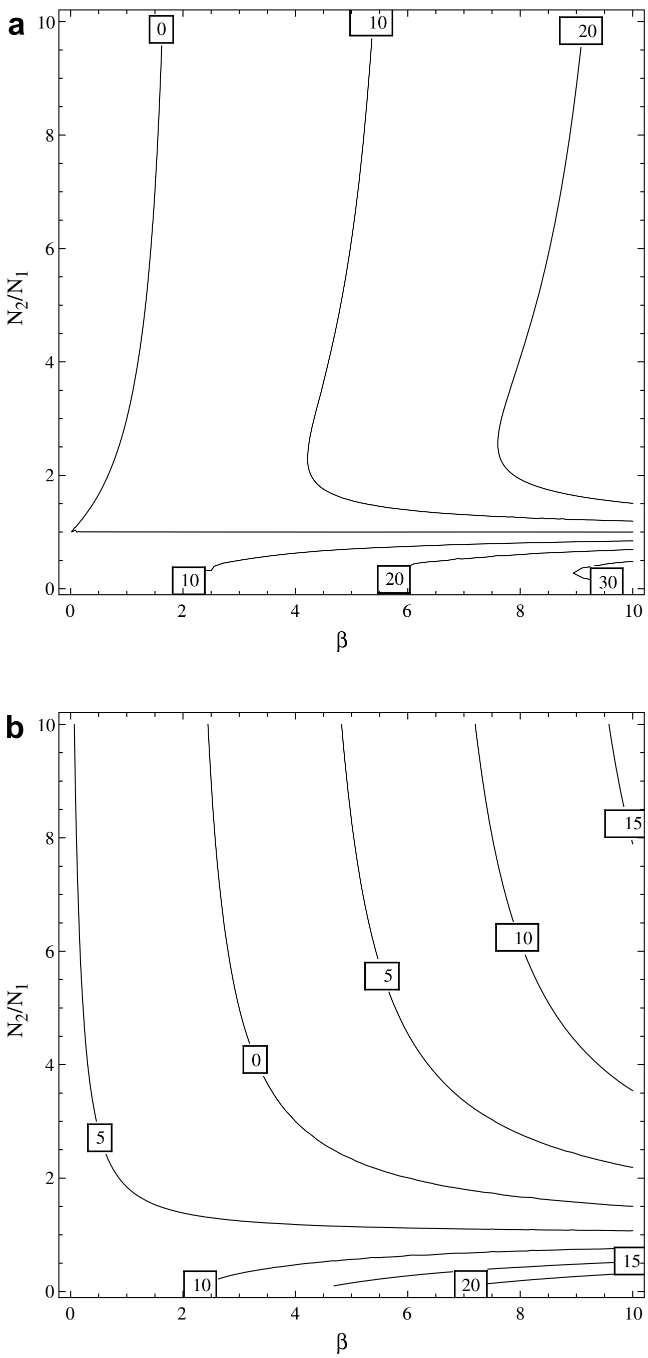

Fig. 2. Contour maps of the Nusselt number Nu, as a function of the asymmetric heating parameter β and the flow asymmetry parameter N_2/N_1, for the Darcy limit. (a) Counterflow (anti-parallel flow) and (b) unidirectional flow (parallel flow), for the case $\xi = 0.5$ (layers of equal thickness).

3.2. Darcy limit

Darcy flow can be interpreted as a flow in clotted arteries/veins. Alternatively, it can be interpreted as a flow in capillary beds. In the case of thermal homogeneity and for layers of equal thickness, we obtain the Nusselt number values displayed in Tables 1a and 1b.

The case $\beta = 0$ corresponds to symmetric heating. The case $N_2/N_1 = 1$ corresponds to symmetric flow. It is obvious that for

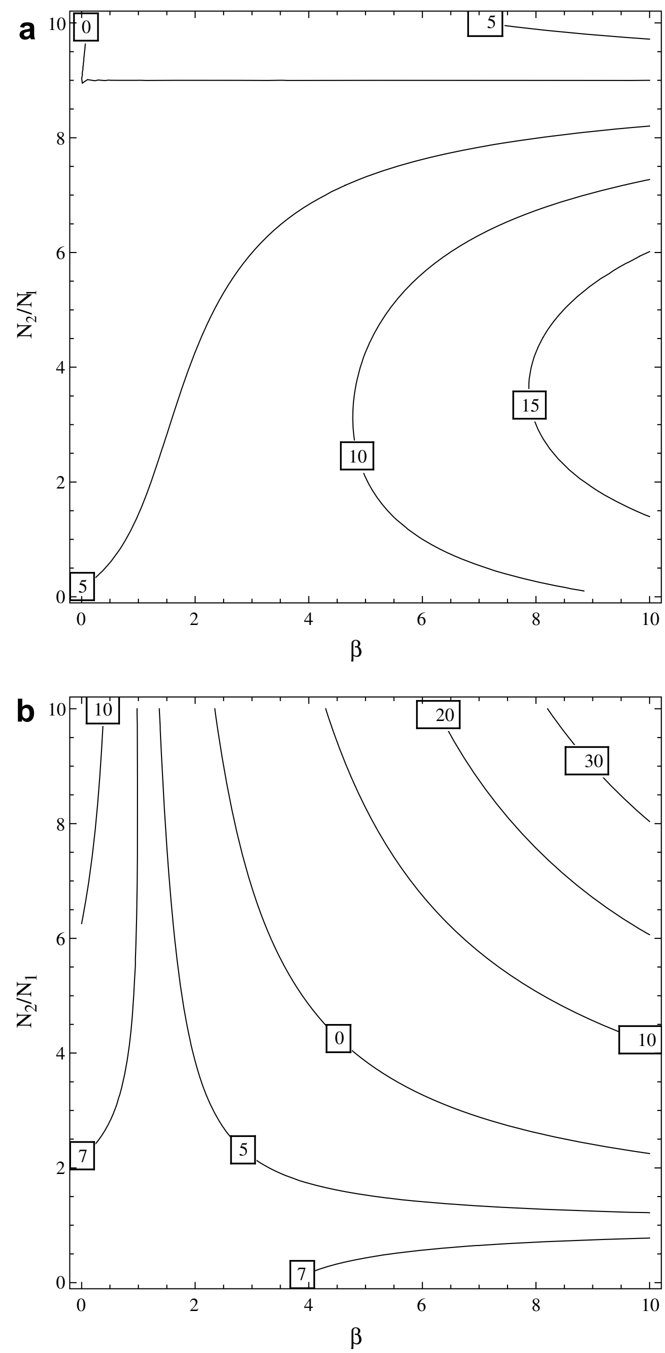

Fig. 3. As for Fig. 1, but now for $\xi = 0.1$.

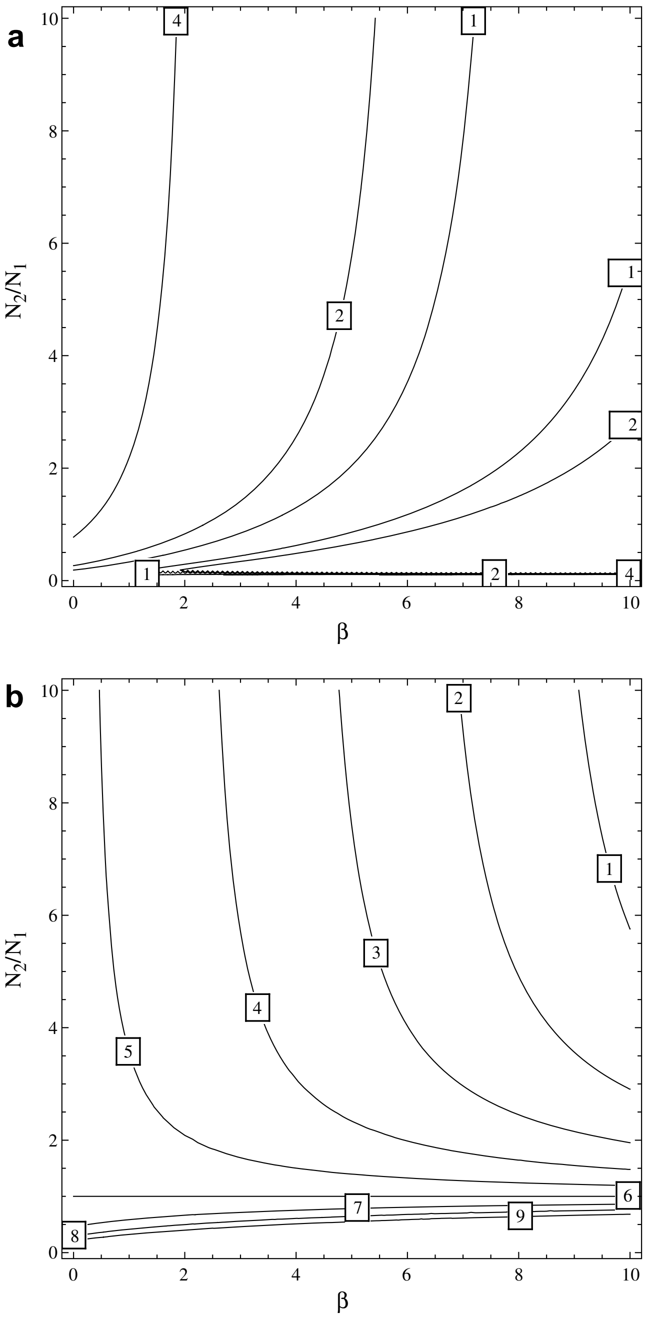

Fig. 4. As for Fig. 1, but now for $\xi = 0.9$.

Table 2a
Values of the Nusselt number *Nu* for counterflow in the clear fluid limit (Eq. (34))

	$M_2/M_1 = 0.1$	1.0	10.0
$\beta = 0$	3.62	0.00	3.62
1	5.83	0.00	1.41
10	25.73	0.00	−18.49

Table 2b
Values of the Nusselt number *Nu* for parallel flow in the clear fluid limit (Eq. (34*))

	$M_2/M_1 = 0.1$	1.0	10.0
$\beta = 0$	4.42	5.38	4.42
1	6.23	5.38	2.61
10	22.49	5.38	−13.66

symmetric flow the value of *Nu* is independent of whether the heating is symmetric or not, the value being zero for counterflow and 6.00 for parallel flow. The effect of increasing β is to increase Nu if $N_2 < N_1$ and to decrease it if $N_2 > N_1$.

The general trends, for the case of layers of equal thickness ($\xi = 0.5$) are illustrated in the contour maps plotted in Fig. 2a and b. There appears to be a singularity as N_2/N_1 tends to zero when β is large. The corresponding contour maps for the cases of $\xi = 0.1$ and $\xi = 0.9$ are shown in Figs. 3a, b and 4a, b respectively.

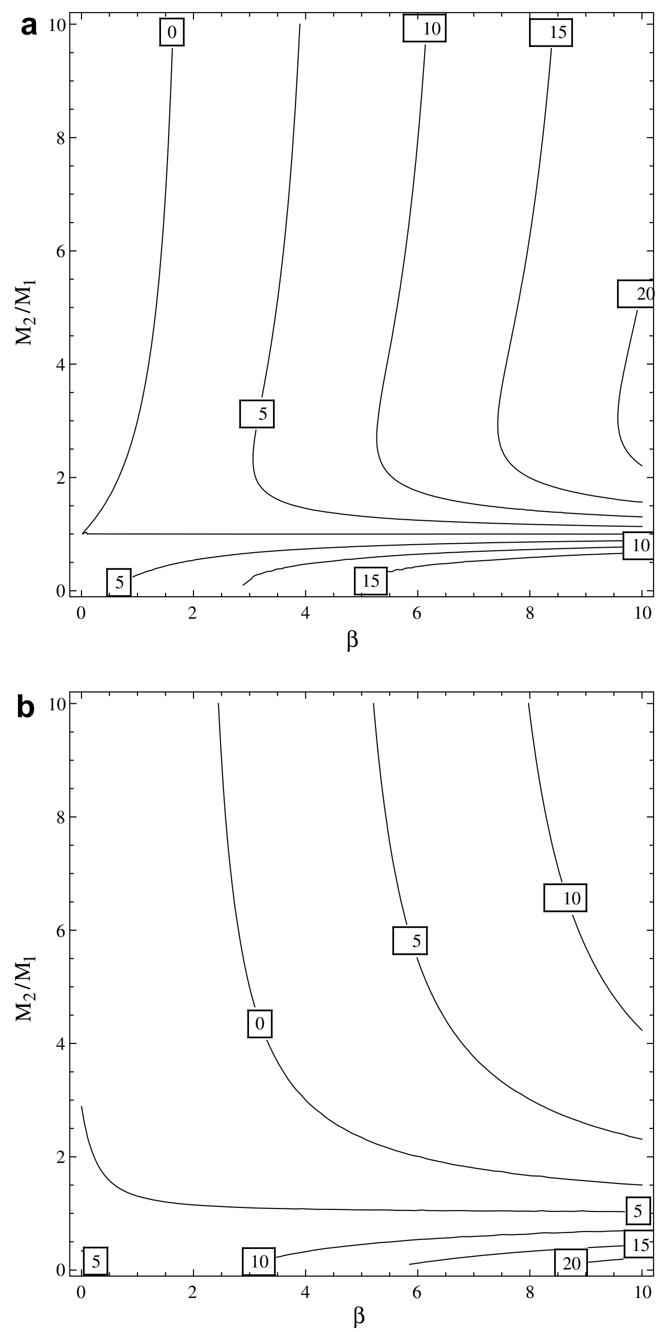

Fig. 5. Contour maps of the Nusselt number *Nu*, as a function of the asymmetric heating parameter β and the flow asymmetry parameter N_2/N_1, for the clear fluid limit. (a) counterflow (anti-parallel flow) and (b) unidirectional flow (parallel flow), for the case $\xi = 0.5$ (layers of equal thickness).

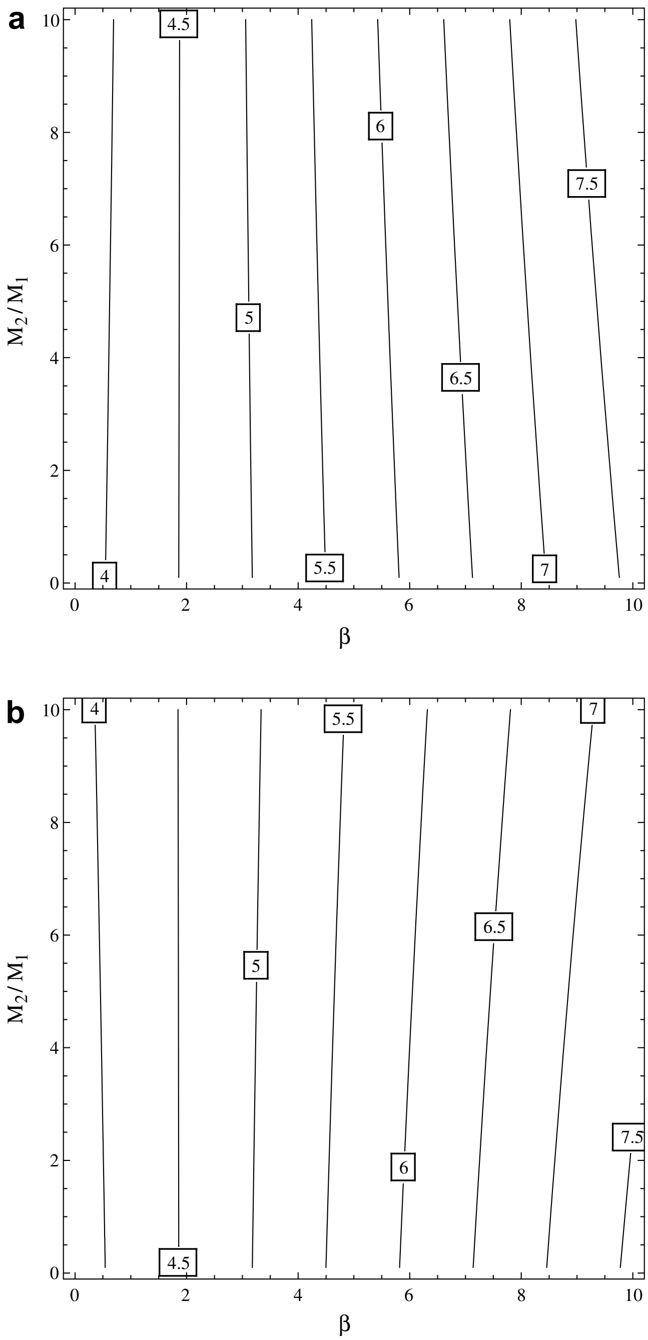

Fig. 6. As for Fig. 1, but now for $\xi = 0.1$.

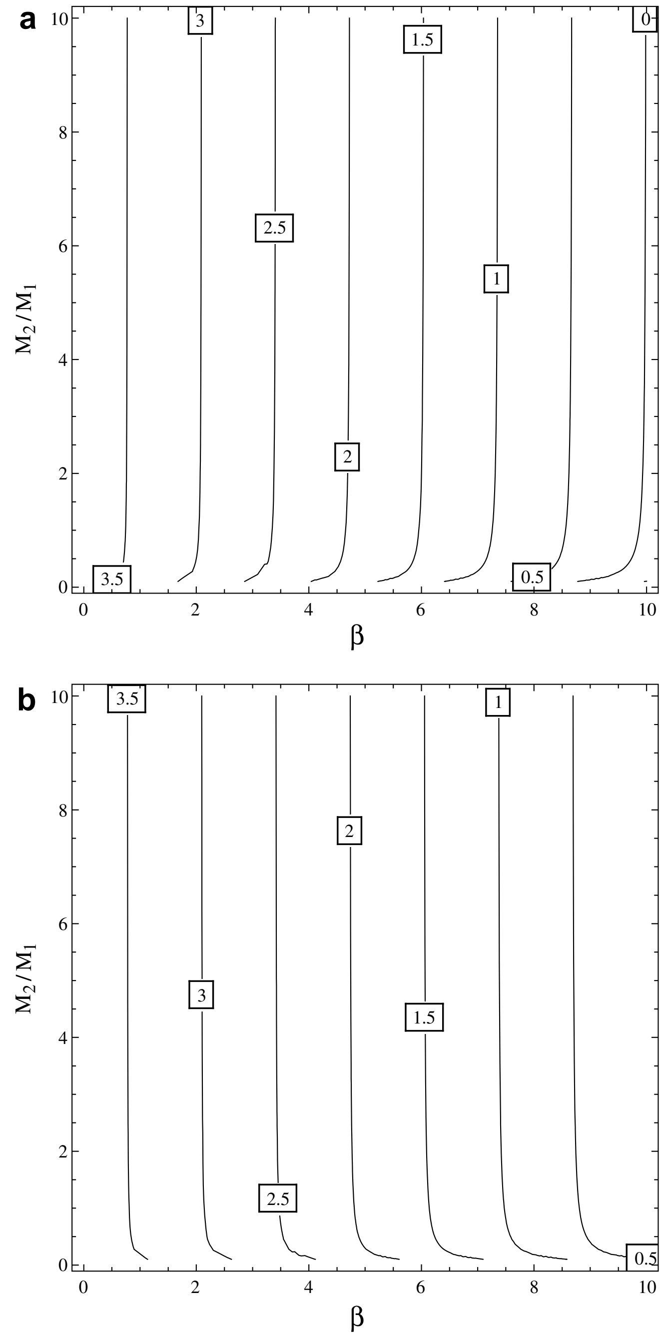

Fig. 7. As for Fig. 1, but now for $\xi = 0.9$.

3.3. Clear fluid limit

The general situation for the clear fluid limit is similar to that for the Darcy limit, but now M_2/M_1 takes the place of N_2/N_1. In the case of thermal homogeneity and for layers of equal thickness, we obtain the Nusselt number values displayed in Tables 2a and 2b.

The case $\beta = 0$ corresponds to symmetric heating. The case $M_2/M_1 = 1$ corresponds to symmetric flow. It is obvious that for symmetric flow the value of *Nu* is independent of whether the heating is symmetric or not, the value being zero for counterflow and 5.38 for parallel flow. The effect of increasing β is to increase *Nu* if $M_2 < M_1$ and to decrease it if $M_2 > M_1$.

The general trends, for the case of layers of equal thickness ($\xi = 0.5$) are illustrated in the contour maps plotted in Fig. 5a and b. There appears to be a singularity as M_2/M_1 tends to zero when β is large. The corresponding contour maps for the cases of $\xi = 0.1$ and $\xi = 0.9$ are shown in Figs. 6a, b and 7a, b respectively. The results for the clear fluid limit are qualitatively similar to those for the Darcy limit.

4. Conclusion

We have obtained an analytical solution for forced convection with counterflow in a parallel plate channel, with asymmetric constant heat-flux boundaries, occupied by a layered saturated porous medium modeled by the Brinkman equation. Detailed results for the Nusselt number have been presented for the cases of the Darcy limit and the clear fluid limit. These provide bounds for values for the general case of a Brinkman porous medium. The dramatic effect of counterflow (in contrast to flow in one direction) is to reduce the value of the Nusselt number, to values that can be negative (and to zero in the case where the velocity profile is symmetric in the sense that the mean velocity is zero). It is expected that the general trends illustrated by the present model will carry over to specific biological situations.

References

[1] A.R.A. Khaled, K. Vafai, The role of porous media in modeling flow and heat transfer in biological tissues, Int. J. Heat Mass Transfer 46 (2003) 4989–5003.

[2] K. Khanafer, K. Vafai, Transport through porous media – a synthesis of the state of the art for the past couple of decades, Ann. Rev. Heat Transfer 14 (2005) 345–383.

[3] K. Khanafer, K. Vafai, The role of porous media in biomedical engineering as related to magnetic resonance imaging and drug delivery, Heat Mass Transfer 42 (2006) 939–953.

[4] A. Nakayama, F. Kuwahara, W. Lui, Macroscopic governing equations for bioheat transfer phenomena, in: Proceedings of the 2nd International Conference on Porous Media and its Application in Science and Engineering, June 17–21, 2007, Kauai, Hawaii, 8 pages.

[5] A. Nakayama, F. Kuwahara, A general bioheat transfer model based on the theory of porous media, Int. J. Heat Mass Transfer 51 (2008) 3179–3189.

[6] D.A. Nield, A.V. Kuznetsov, Effects of heterogeneity in forced convection in a porous medium: parallel plate channel, asymmetric property variation, and asymmetric heating, J. Porous Media 4 (2001) 137–148.